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Bernoulli polynomials and asymptotic expansions of the quotient of gamma functions

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ABSTRACT

The main subject of this paper is the analysis of asymptotic expansions of Wallis quotient function $\frac{\Gamma(x+t)}{\Gamma(x+s)}$ and Wallis power function $\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}$, when x tends to infinity. Coefficients of these expansions are polynomials derived from Bernoulli polynomials. The key to our approach is the introduction of two intrinsic variables $\alpha = \frac{1}{2}(t+s-1)$ and $\beta = \frac{1}{4}(1+t-s)(1-t+s)$ which are naturally connected with Bernoulli polynomials and Wallis functions. Asymptotic expansion of Wallis functions in terms of variables t and s and also α and β is given. Application of the new method leads to the improvement of many known approximation formulas of the Stirling's type.

1. Introduction

The quotient of two gamma functions is connected to the Wallis product on the right:

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} = \frac{1}{\sqrt{\pi}}\frac{(2n)!!}{(2n-1)!!},$$

so we shall use the name *Wallis quotient function* for the more general quotient

$$W(x,t,s) := \frac{\Gamma(x+t)}{\Gamma(x+s)}. \tag{1.1}$$

This quotient, especially in the form $\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}$ has been investigated by many authors in various papers; see [1–8] and the literature cited therein. The main subject is the asymptotic behavior of the function W as x tends to infinity and inequalities for finite part of such expansions.

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The logarithm of gamma function has asymptotic expansion in complex domain as $x \to \infty$, in terms of the Bernoulli polynomials [9, p. 32]:

$$\log \Gamma(x+t) = \left(x+t-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{k=1}^{n} \frac{(-1)^{k+1}B_{k+1}(t)}{k(k+1)}x^{-k} + O(x^{-n-1}), \tag{1.2}$$

uniformly on $|\arg x| \le \pi - \varepsilon$, $\varepsilon > 0$ being given in advance. It is usual to write this expansion in the form

$$\log \Gamma(x+t) \sim \left(x+t-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}B_{n+1}(t)}{n(n+1)}x^{-n}. \tag{1.3}$$

This notation of asymptotic expansion will be used in the rest of the paper.

The digamma (psi) function has the following expansion [9, p. 32]:

$$\psi(x+t) \sim \log x - \sum_{n=1}^{\infty} \frac{(-1)^n B_n(t)}{n} x^{-n}, \tag{1.4}$$

which can be obtained from (1.3) by termwise differentiation.

The ratio of two gamma functions of (not very different) large arguments gives a well-known asymptotic expansion, see [9, p. 33]:

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim x^{t-s} \sum_{n=0}^{\infty} (-1)^n \frac{(s-t)_n}{n!} B_n^{(t-s+1)}(t) \frac{1}{x^n}, \tag{1.5}$$

where the symbol $B_n^{(a)}(t)$ stands for the generalized Bernoulli polynomials, defined by the following generating function:

$$\frac{x^a e^{tx}}{(e^x-1)^a} = \sum_{n=0}^{\infty} B_n^{(a)}(t) \frac{x^n}{n!}, \tag{1.6}$$

and $(t)_n$ is Pochhammer's symbol defined as product $t(t+1)\cdots(t+n-1)$. This expansion is analyzed in [1]. Note that Bernoulli polynomials (B_n) are defined by (1.6) for $a = 1$.

In [3], authors studied properties of the function

$$F(x,t,s) := \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{\frac{1}{t-s}}. \tag{1.7}$$

We shall call it here *the Wallis power function*. Among other things, it is proved that the function $x \mapsto F(x,t,s)$ is convex for $|t-s| < 1$ and concave for $|t-s| > 1$.

In a recent paper [5], Mortici has proved the following approximation of the quotient

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt{n + \frac{1}{4} + \frac{1}{32n} - \frac{1}{128n^2} - \frac{5}{2048n^3} + \frac{23}{8192n^4}}. \tag{1.8}$$

In other words, this is the asymptotic expansion of the function $F\left(x, 1, \frac{1}{2}\right)$. We shall use approximation sign when displaying formulas with finitely many terms, suggesting numerical approximation obtained by formula.

In [10], using same techniques, authors proved the following expansion

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{\frac{1}{t-s}} \sim x + a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3} + \frac{e}{x^4} \tag{1.9}$$

where

$$\begin{aligned}
a &= \frac{s+t-1}{2}, \\
b &= \frac{1}{24}[1-(t-s)^2], \\
c &= -ab, \\
d &= \frac{b}{10}\left(10a^2 - 13b - 1\right), \\
e &= -\frac{ab}{10}\left(10a^2 - 39b - 3\right).
\end{aligned}$$

This expansion gives (1.8) for $t = 1,\ s = \frac{1}{2}$.

The main purpose of this paper is to identify polynomials in the asymptotic expansions of Wallis function and Wallis power function. It will be shown that the function $F(x,t,s)$ has more natural asymptotic expansion compared to the quotient function $W(x,t,s)$. We shall deduce general formulas for the expansion (1.9) and present efficient algorithm for calculating this asymptotic expansion.

The paper is organized as follows. In the second section basic formula for asymptotic expansion of the Wallis power function in terms of variables t and s is derived. Properties of the Bernoulli polynomials and the Bernoulli quotient function will be analyzed in Sections 3 and 4. In Section 5, properties of polynomials which are coefficients in asymptotic expansion of the function F are discussed. Asymptotic expansion of the Wallis quotient is studied in Section 6 and another approach through shifted variable is analyzed in Section 7. Finally, applications of obtained formulas are given in Section 8.

2. First asymptotic expansion and motivation

We state the following theorem.

Theorem 2.1. *It holds*

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{\frac{1}{t-s}} \sim \sum_{n=0}^{\infty} P_n(t,s)x^{-n+1} \tag{2.1}$$

where $P_n(t,s)$ are polynomials of order n defined by

$$\begin{aligned} &P_0(t,s) = 1, \\ &P_n(t,s) = \frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\frac{B_{k+1}(t)-B_{k+1}(s)}{(k+1)(t-s)}P_{n-k}(t,s), \quad n \geq 1. \end{aligned} \tag{2.2}$$

Here $B_k(t)$ stands for the Bernoulli polynomials.

The first few polynomials (P_n) are

$$\begin{aligned} P_1 &= \frac{1}{2}(t+s-1) \\ P_2 &= \frac{1}{24}(1-t^2+2ts-s^2) \\ P_3 &= \frac{1}{48}(1-t-s-t^2+2ts-s^2+t^3-t^2s-ts^2+s^3) \\ P_4 &= \frac{1}{5760}(23-120t-120s+50t^2+140st+50s^2+120t^3-120st^2-120s^2t+120s^3 \\ &\quad -73t^4+52t^3s+42t^2s^2+52ts^3-73s^4). \end{aligned} \tag{2.3}$$

Further calculation is very inconvenient. Introducing intrinsic variables α and β, we shall obtain much simpler expressions for these polynomials; see (5.3).

The asymptotic expansion

$$f(x) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \quad \text{as } x \to \infty \tag{2.4}$$

is called *asymptotic power series*. The manipulations with asymptotic series in the proof of this theorem and in the rest of the paper are justified by properties of asymptotic power series; see [11, Section 1.6]. It is known that two such expansions can be added or multiplied, and also divided provided that leading coefficient of denominator is different from zero. Also, asymptotic power series may be substituted in finite linear combinations, in polynomials, and in asymptotic power series. Coefficients of the new expansion are obtained by formal substitution and rearrangement of terms.

We extract results which will be used in the following.

Lemma 2.2 *([11, p. 20]). If the function g has expansion into power series*

$$g(x) = \sum_{n=0}^{\infty} c_n x^n, \quad \text{as } x \to 0$$

and $f(x)$ has asymptotic expansion (2.4) with leading coefficient $a_0 = 0$, then $g(f(x))$ has asymptotic expansion whose coefficients can be obtained by formal substitution and rearrangement of terms.

Corollary 2.3. *The Wallis power function* $\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}$ *has asymptotic expansion as* $x \to \infty$.

Proof. We can use known expansion of the logarithm of gamma function (1.3) to obtain

$$\frac{1}{t-s}\log\frac{\Gamma(x+t)}{\Gamma(x+s)} - \log x \sim \sum_{n=1}^{\infty} a_n x^{-n}.$$

Applying previous lemma to the function $g(x) = \exp(x)$, we have

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \sim x\sum_{n=0}^{\infty} b_n x^{-n} \tag{2.5}$$

and the required expansion is ensured. □

We shall also recall the main results about differentiating of asymptotic series.

Lemma 2.4 ([11, p. 21]). *If the function* f *in* (2.4) *is differentiable and if* f' *possesses an asymptotic power series expansion, then*

$$f'(x) \sim -\frac{a_1}{x^2} - \frac{2a_2}{x^3} - \frac{3a_3}{x^4} - \cdots, \quad \text{as } x \to \infty. \tag{2.6}$$

Lemma 2.5 ([11, p. 21]). *Let* R_1 *be the region* $|x| > r_1, \gamma_1 < \arg x < \gamma_2$, *let* $r_2 > r_1, \gamma_1 < \gamma_1' < \gamma_2' < \gamma_2$ *and let* R_2 *be the region* $|x| > r_2, \gamma_1' < \arg x < \gamma_2'$. *If* $f(x)$ *is regular in* R_1 *and*

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}$$

holds uniformly in $\arg x$ *as* $x \to \infty$ *in* R_1, *then*

$$f'(x) \sim \sum_{n=1}^{\infty} (-n) a_n x^{-n-1}$$

holds uniformly in $\arg x$ *as* $x \to \infty$ *in* R_2.

Proof of Theorem 2.1. By Corollary 2.3, function $F(x,t,s)$ has asymptotic expansion of the form (2.5), where coefficients b_n depend on t and s, therefore, representation (2.1) is valid, with $P_0(t,s) = 1$. Further, it holds

$$\frac{\partial}{\partial x}F(x,t,s) = \frac{1}{t-s}F(x,t,s)[\psi(x+t) - \psi(x+s)].$$

Since both functions $F(x,t,s)$ and $[\psi(x+t) - \psi(x+s)]$ have asymptotic expansions as $x \to \infty$, there exists asymptotic expansion for their product and, by Lemma 2.4, it can be obtained by termwise differentiation of expansion (2.1). Therefore, it holds

$$\frac{1}{t-s}F(x,t,s)[\psi(x+t) - \psi(x+s)] \sim \sum_{n=0}^{\infty} P_n(t,s)\frac{-n+1}{x^n}. \tag{2.7}$$

Using asymptotic expansion of the psi function, see (1.4), we obtain

$$\left[\sum_{j=0}^{\infty} P_j(t,s)\frac{1}{x^{j-1}}\right]\left[\sum_{k=1}^{\infty}\frac{(-1)^{k+1}[B_k(t) - B_k(s)]}{k(t-s)}\frac{1}{x^k}\right] \sim \sum_{n=0}^{\infty} P_n(t,s)\frac{-n+1}{x^n}.$$

After rearrangement of the terms of the product on the left-hand side and comparing the coefficients next to x^{-n} we get

$$\sum_{k=0}^{n}(-1)^k\frac{B_{k+1}(t) - B_{k+1}(s)}{(k+1)(t-s)}P_{n-k}(t,s) = -(n-1)P_n(t,s).$$

The member of the sum for $k = 0$ has the value

$$\frac{B_1(t) - B_1(s)}{t-s}P_n(t,s) = P_n(t,s).$$

Hence,

$$nP_n(t,s)=\sum_{k=1}^{n}(-1)^{k+1}\frac{B_{k+1}(t)-B_{k+1}(s)}{(k+1)(t-s)}P_{n-k}(t,s)$$

which proves the theorem. $\square$

Analyzing values (2.3), we see that given values P_n for $n=1,2,3,4$ can be written as a function of variables:

$$\alpha=\frac{t+s-1}{2},\qquad \beta=\frac{1}{4}[1-(t-s)^2] \tag{2.8}$$

as stated in formula (1.9). We shall show that all polynomials $P_n(t,s)$ can be expressed in terms of these variables. This expansion will have the following form

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{\frac{1}{t-s}}\sim x+\sum_{n=0}^{\infty}Q_{n+1}(\alpha,\beta)\frac{1}{x^n} \tag{2.9}$$

where $Q_n(\alpha,\beta)$ is a polynomial obtained from $P_n(t,s)$ by the change of variables.

Let us explain the role of the variables α and β in asymptotic expansion from another point of view.

The variable $\alpha=\frac{1}{2}(t+s-1)$ appears naturally since it is a limit of the function $F(x,t,s)-x$ as x tends to infinity.

If $t=s+1$, formula (2.1) has only two terms, namely

$$F(x,s+1,s)=x+s.$$

The same expression holds if $t=s-1$, again we have $F(x,s-1,s)=x+s$. Therefore, polynomials $P_n(s+1,s)$ and $P_n(s-1,s)$ are identically equal to zero for $n\geq 2$.

Let us denote in the rest of this paper

$$\beta_1=\frac{1}{2}(1+t-s),\qquad \beta_2=\frac{1}{2}(1-t+s). \tag{2.10}$$

We shall prove that polynomial $P_n(t,s)$ is divisible by

$$\beta=\beta_1\beta_2=\frac{1}{4}[1-(t-s)^2] \tag{2.11}$$

for all $n\geq 2$.

Further, it will be proved that $P_n(t,s)$ is a function of variables α and β, which will justify expansion (2.9).

3. Bernoulli polynomials

For the convenience of the reader, we list the first few polynomials:

$$\begin{aligned}
B_0(x)&=1\\
B_1(x)&=x-\frac{1}{2}\\
B_2(x)&=x^2-x+\frac{1}{6}\\
B_3(x)&=x^3-\frac{3}{2}x^2+\frac{1}{2}x\\
B_4(x)&=x^4-2x^3+x^2-\frac{1}{30}\\
B_5(x)&=x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{1}{6}x\\
B_6(x)&=x^6-3x^5+\frac{5}{2}x^4-\frac{1}{2}x^2+\frac{1}{42}.
\end{aligned}$$

Let us denote

$$v:=x-\frac{1}{2}. \tag{3.1}$$

Then these polynomials have another expression through variable v. For example

$$B_5(x) = v\left(v^4 - \frac{5}{6}v^2 + \frac{7}{48}\right)$$
$$B_6(x) = v^6 - \frac{5}{4}v^4 + \frac{7}{16}v^2 - \frac{31}{1344}.$$

It will be proved that the Bernoulli polynomials can be expressed through polynomials $(C_n(v))$ and $(D_n(v))$ of order n such that it holds

$$B_{2n}(x) = C_n(v^2), \qquad B_{2n+1}(x) = vD_n(v^2). \tag{3.2}$$

The first few polynomials (C_n and D_n) are

$$C_0 = 1$$
$$C_1 = v - \frac{1}{12}$$
$$C_2 = v^2 - \frac{1}{2}v + \frac{7}{240}$$
$$C_3 = v^3 - \frac{5}{4}v^2 + \frac{7}{16}v - \frac{31}{1344}$$
$$C_4 = v^4 - \frac{7}{3}v^3 + \frac{49}{24}v^2 - \frac{31}{48}v + \frac{127}{3840}$$
$$C_5 = v^5 - \frac{15}{4}v^4 + \frac{49}{8}v^3 - \frac{155}{32}v^2 + \frac{381}{256}v - \frac{2555}{33\,792}$$

and

$$D_0 = 1$$
$$D_1 = v - \frac{1}{4}$$
$$D_2 = v^2 - \frac{5}{6}v + \frac{7}{48}$$
$$D_3 = v^3 - \frac{7}{4}v^2 + \frac{49}{48}v - \frac{31}{192}$$
$$D_4 = v^4 - 3v^3 + \frac{147}{40}v^2 - \frac{31}{16}v + \frac{381}{1280}$$
$$D_5 = v^5 - \frac{55}{12}v^4 + \frac{77}{8}v^3 - \frac{341}{32}v^2 + \frac{1397}{256}v - \frac{2555}{3072}.$$

Theorem 3.1. *Let $(B_n(t))$ be a sequence of the Bernoulli polynomials. Denote $v = \left(t - \frac{1}{2}\right)^2$. Then there exist polynomials $(C_n(v))$ and $(D_n(v))$ of degree n such that (3.2) holds.*

These polynomials satisfy relations

$$C_n'(v) = nD_{n-1}(v), \tag{3.3}$$
$$D_n(v) + 2vD_n'(v) = (2n+1)C_n(v) \tag{3.4}$$

and can be calculated by

$$C_n(v) = n\int_0^v D_{n-1}(v)\mathrm{d}v + \delta_n, \tag{3.5}$$
$$D_n(v) = \frac{2n+1}{2\sqrt{v}}\int_0^v \frac{C_n(v)}{\sqrt{v}}\mathrm{d}v, \tag{3.6}$$

where

$$\delta_n = -(1 - 2^{1-2n})B_{2n} \tag{3.7}$$

and B_{2n} are the Bernoulli numbers.

Proof. First, let us prove the existence of the polynomials C_n and D_n. Define new polynomials $\widetilde{B}_n$ by the formula

$$\widetilde{B}_n(v) := B_n(x), \quad v = x - \frac{1}{2}.$$

Then, using the symmetry property of the Bernoulli polynomials, we obtain

$$\widetilde{B}_n(-v) = \widetilde{B}_n\left(\frac{1}{2} - x\right) = B_n(1-x) = (-1)^n B_n(x) = (-1)^n \widetilde{B}_n(v).$$

Therefore, $\widetilde{B}_n$ is even function for even n and odd function for odd n. Hence, there exist polynomials C_n and D_n such that (3.2) holds.

Let us prove (3.3) and (3.4). Using (3.2) and the property of the Bernoulli polynomials, it holds

$$B'_{2n}(x) = C'_n(v^2) \cdot 2v = 2n \qquad B_{2n-1}(x) = 2nvD_{n-1}(v^2)$$

wherefrom it follows

$$C'_n(u) = nD_{n-1}(u).$$

Similarly,

$$B'_{2n+1}(x) = D_n(v^2) + vD'_n(v^2) \cdot 2v = (2n+1)C_n(v^2).$$

Hence,

$$D_n(u) + 2uD'_n(u) = (2n+1)C_n(u).$$

Relation (3.5) is equivalent to (3.3). We shall prove (3.7). By the definition of the sequence (δ_n), we have

$$\delta_n = C_n(0) = B_{2n}\left(\frac{1}{2}\right) = -(1-2^{1-2n})B_{2n},$$

where B_{2n} are the Bernoulli numbers; see [12, p. 805].

From (3.4) it follows

$$D_n(v) = \frac{2n+1}{2\sqrt{v}} \int_0^v \frac{C_n(v)}{\sqrt{v}} \mathrm{d}v + K_n,$$

where K_n is a constant which should be determined. Taking a limit, we get

$$\lim_{v \to 0} D_n(v) = (2n+1)C_n(0) + K_n.$$

Since D_n and C_n are polynomials, (3.4) implies

$$D_n(0) = (2n+1)C_n(0),$$

therefore, it holds $K_n = 0$ and the theorem is proved. $\square$

Variables β_1 and β_2 satisfy $\beta_1 + \beta_2 = 1$, therefore the product $\beta = \beta_1\beta_2$ is equal to $\beta_1 - \beta_1^2 = \frac{1}{4} - \left(\beta_1 - \frac{1}{2}\right)^2$. It means that polynomials $(C_n(\beta_1))$ and $(D_n(\beta_1))$ can be written as polynomials of variable β. The following sequence (G_n) will be essential in the future:

$$G_n(\beta) := D_n\left(\frac{1}{4} - \beta\right). \tag{3.8}$$

The first few polynomials (G_n) are

$$\begin{aligned}
G_0 &= 1\\
G_1 &= -\beta\\
G_2 &= \beta^2 + \frac{1}{3}\beta\\
G_3 &= -\beta^3 - \beta^2 - \frac{1}{3}\beta\\
G_4 &= \beta^4 + 2\beta^3 + \frac{9}{5}\beta^2 + \frac{3}{5}\beta\\
G_5 &= -\beta^5 - \frac{10}{3}\beta^4 - \frac{17}{3}\beta^3 - 5\beta^2 - \frac{5}{3}\beta\\
G_6 &= \beta^6 + 5\beta^5 + \frac{41}{3}\beta^4 + \frac{472}{21}\beta^3 + \frac{691}{35}\beta^2 + \frac{691}{105}\beta.
\end{aligned}$$

Compared with D_n, polynomial G_n has much simpler form, which indicates that the choice of variable β is more natural for the expression of Bernoulli polynomials than variable υ.

Theorem 3.2. *It holds*

$$B_{2n+1}(\beta_1) = \frac{1}{2}(t-s)G_n(\beta). \tag{3.9}$$

Proof. Theorem follows from (3.2) and (3.8), since

$$\upsilon = \beta_1 - \frac{1}{2} = \frac{1}{2}(t-s). \quad \square$$

4. Bernoulli quotient function

Quotient

$$\Delta_n := \frac{B_{n+1}(t) - B_{n+1}(s)}{(n+1)(t-s)}, \quad n \geq 0 \tag{4.1}$$

will be called *the Bernoulli quotient function*. It plays a key role in investigating asymptotic expansion (2.9), because of formula (2.2).

Here are the first few polynomials (Δ_n):

$$\begin{aligned}
\Delta_0 &= 1 \\
\Delta_1 &= \frac{1}{2}(s+t-1) \\
\Delta_2 &= \frac{1}{6}(1 - 3s + 2s^2 - 3t + 2st + 2t^2) \\
\Delta_3 &= \frac{1}{4}(s - 2s^2 + s^3 + t - 2st + s^2t - 2t^2 + st^2 + t^3) \\
\Delta_4 &= \frac{1}{30}(-1 + 10s^2 - 15s^3 + 6s^4 + 10st - 15s^2t + 6s^3t + 10t^2 - 15st^2 + 6s^2t^2 - 15t^3 + 6st^3 + 6t^4).
\end{aligned}$$

Further calculation is inconvenient. It will be shown that Δ_n can be written in terms of variables α and β. Let

$$\nabla_n(\alpha, \beta) := \Delta_n(t, s).$$

An efficient algorithm for calculating the Bernoulli quotient function through the sequence (∇_n) will be given.

Let

$$T_n = \beta_1^n + \beta_1^{n-1}\beta_2 + \cdots + \beta_1\beta_2^{n-1} + \beta_2^n$$

where β_1 and β_2 are defined as in (2.10).

Lemma 4.1. *T_n is a polynomial in variable β. Let $T_0 = 1$. It holds*

$$\begin{aligned}
T_1 &= 1 \\
T_n &= T_{n-1} - \beta T_{n-2}, \quad n \geq 2.
\end{aligned} \tag{4.2}$$

Proof. The assertion of the lemma follows from the identity

$$(\beta_1 + \beta_2)T_{n-1} = T_n + \beta_1\beta_2 T_{n-2}$$

since $\beta_1 + \beta_2 = 1$. $\square$

From relation (4.2) one can obtain explicit formula for T_n:

$$T_n = \frac{1}{2^{n+1}\sqrt{1-4\beta}}\left[(1+\sqrt{1-4\beta})^{n+1} - (1-\sqrt{1-4\beta})^{n+1}\right]$$

but it is not of practical use.

The first few polynomials (T_n) are

$$\begin{aligned}
T_0 &= 1\\
T_1 &= 1\\
T_2 &= 1-\beta\\
T_3 &= 1-2\beta\\
T_4 &= 1-3\beta+\beta^2\\
T_5 &= 1-4\beta+3\beta^2\\
T_6 &= 1-5\beta+6\beta^2-\beta^3.
\end{aligned} \tag{4.3}$$

Theorem 4.2. *∇_n is a polynomial in variables α and β and can be expressed as*

$$\nabla_n(\alpha,\beta) = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}B_{n-k}(\alpha)T_k(\beta) \tag{4.4}$$

for all $n \geq 0$.

Proof. From generating function of Bernoulli polynomials it follows:

$$\frac{xe^{tx}-xe^{sx}}{e^x-1} = \sum_{n=0}^{\infty}\frac{B_n(t)-B_n(s)}{n!}x^n.$$

Function on the left-hand side can be written as

$$\frac{xe^{\alpha x}}{e^x-1}[e^{x\beta_1}-e^{x\beta_2}] \tag{4.5}$$

where α, β_1 and β_2 are defined as before.

Now, we have

$$\left[\sum_{k=0}^{\infty}B_k(\alpha)\frac{x^k}{k!}\right]\left[\sum_{j=0}^{\infty}(\beta_1^j-\beta_2^j)\frac{x^j}{j!}\right] = \sum_{n=0}^{\infty}\frac{B_n(t)-B_n(s)}{n!}x^n.$$

It follows

$$\frac{B_n(t)-B_n(s)}{t-s} = \sum_{k=1}^{n}\binom{n}{k}B_{n-k}(\alpha)\frac{\beta_1^k-\beta_2^k}{t-s}.$$

Since $t-s=\beta_1-\beta_2$, it holds

$$\frac{\beta_1^k-\beta_2^k}{t-s} = T_{k-1}$$

and this proves the theorem. □

Let us describe an algorithm for calculating sequence (∇_n) which is more efficient than (4.4). The first few polynomials (∇_n) are

$$\begin{aligned}
\nabla_0 &= 1\\
\nabla_1 &= \alpha\\
\nabla_2 &= \alpha^2-\frac{1}{3}\beta\\
\nabla_3 &= \alpha^3-\alpha\beta\\
\nabla_4 &= \alpha^4-2\alpha^2\beta+\frac{1}{15}\beta+\frac{1}{5}\beta^2\\
\nabla_5 &= \alpha^5-\frac{10}{3}\alpha^3\beta+\alpha\beta^2+\frac{1}{3}\alpha\beta\\
\nabla_6 &= \alpha^6-5\alpha^4\beta+\alpha^2\beta+3\alpha^2\beta^2-\frac{1}{21}\beta-\frac{1}{7}\beta^2-\frac{1}{7}\beta^3.
\end{aligned} \tag{4.6}$$

In this sequence, we recognize that (∇_n) forms an Appell sequence on the first argument, i.e., the following relation holds.

Theorem 4.3. *The sequence* (∇_n) *satisfies relation*

$$\frac{\partial \nabla_n}{\partial \alpha} = n\nabla_{n-1} \tag{4.7}$$

for all $n \geq 1$.

Proof. Calculating a partial derivative of (4.4) we obtain:

$$\begin{aligned}\frac{\partial \nabla_n}{\partial \alpha} &= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}B'_{n-k}(\alpha)T_k(\beta)\\ &= \frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k+1}(n-k)B_{n-k-1}(\alpha)T_k(\beta)\\ &= \sum_{k=0}^{n-1}\binom{n}{k+1}B_{n-k-1}(\alpha)T_k(\beta)\\ &= n\nabla_{n-1}. \quad \square\end{aligned}$$

Immediate consequence of the previous theorem is the following relation

$$\nabla_n(\alpha, \beta) = n\int_0^{\alpha} \nabla_{n-1}(\alpha, \beta)\mathrm{d}\alpha + \nabla_n(0, \beta). \tag{4.8}$$

Therefore, to efficiently reconstruct sequence (4.6), besides relation (4.8), we also need to know the constant of integration which is a function of argument β.

Theorem 4.4. *It holds*

$$\nabla_{2n+1}(0, \beta) = 0, \tag{4.9}$$

$$\nabla_{2n}(0, \beta) = \frac{1}{2n+1}G_n(\beta), \tag{4.10}$$

where (G_n) *is defined in* (3.8)*, and it follows*

$$G_n(\beta) = \sum_{k=0}^{2n}\binom{2n+1}{k+1}B_{2n-k}T_k(\beta). \tag{4.11}$$

Proof. Since $\alpha = 0$ is equivalent to $s + t = 1$, we can write

$$\nabla_n(0, \beta) = \frac{B_{n+1}(t) - B_{n+1}(1-t)}{(n+1)(2t-1)}.$$

Statement (4.9) follows from the property of the Bernoulli polynomials

$$B_n(1-t) = (-1)^n B_n(t).$$

If $\alpha = 0$, it follows that $\beta_1 = t$, $\beta_2 = s$. From Theorem 3.2

$$\begin{aligned}\nabla_{2n}(0, \beta) &= \frac{B_{2n+1}(\beta_1) - B_{2n+1}(\beta_2)}{(2n+1)(t-s)}\\ &= \frac{2B_{2n+1}(\beta_1)}{(2n+1)(t-s)} = \frac{1}{2n+1}G_n(\beta)\end{aligned}$$

and relation (4.11) follows directly from (4.4). $\square$

The summation in (4.11) can be shortened.

Theorem 4.5. *The sequence* (G_n) *is given by the following algorithm:*

$$\begin{aligned}&d_{n+1} = d_{n+2} = 0\\ &d_k = \binom{2n+1}{k+1}B_{2n-k} + d_{k+1} - \beta d_{k+2}, \quad k \in \{n, n-1, \ldots, 0\}\\ &G_n = d_0\end{aligned}$$

where B_n *are Bernoulli numbers.*

Proof. The proof is based on the fast summation algorithm (see [13, p. 476]) applied to the formula (4.11), since (T_n) satisfies recurrence relation (4.2). For a sequence (p_j) which satisfies recursive relation

$$p_{j+1}(x) + \alpha_j(x)p_j(x) + \beta_j(x)p_{j-1}(x) = 0, \quad j \geq 2,$$

the sum

$$f_n(x) = \sum_{j=0}^{n} a_j p_j(x)$$

can be calculated by the algorithm

$$\begin{aligned}
&d_{n+1} = d_{n+2} = 0\\
&d_j = a_j - \alpha_j(x)d_{j+1} - \beta_{j+1}(x)d_{j+2}\\
&f_n(x) = d_0p_0(x) + d_1p_1(x) + \alpha_0(x)d_1(x)p_0(x).
\end{aligned}$$

Proof of the theorem follows since $\alpha_j(x) = -1$, $\beta_j(x) = \beta$ for all j. □

Finally, another approach for calculating sequence (∇_n) can be derived.

Theorem 4.6. *It holds*

$$\frac{B_n(t) - B_n(s)}{t - s} = \sum_{k=0}^{n} \binom{n}{k} \frac{B_k(\beta_1) - B_k(\beta_2)}{\beta_1 - \beta_2} \alpha^{n-k}. \tag{4.12}$$

Therefore

$$\nabla_n(\alpha, \beta) = \frac{1}{n+1} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} G_k(\beta)\alpha^{n-2k}. \tag{4.13}$$

Proof. The factors in the expression (4.5) can be changed in a way

$$e^{\alpha x} \frac{xe^{x\beta_1} - xe^{x\beta_2}}{e^x - 1} = \frac{xe^{tx} - xe^{sx}}{e^x - 1}.$$

Hence, it holds

$$\left[\sum_{j=0}^{\infty} \frac{\alpha^j}{j!} x^j\right] \left[\sum_{k=0}^{\infty} \frac{B_k(\beta_1) - B_k(\beta_2)}{k!} x^k\right] = \sum_{n=0}^{\infty} \frac{B_n(t) - B_n(s)}{n!} x^n.$$

The rest of the proof follows as in Theorems 4.2 and 3.2. □

5. Algorithm for calculating polynomials Q_n

According to the formulas (2.2) and (4.4), in the expansion

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \sim x + \sum_{n=0}^{\infty} P_{n+1}(t, s)x^{-n}$$

the coefficients can be written in a way:

$$P_n(t, s) = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} \nabla_k(\alpha, \beta) P_{n-k}(t, s).$$

Since $P_0(t, s) = 1$, it follows by induction that $P_n(t, s)$ is a function of arguments α and β. Therefore, polynomials Q_n are defined such that it holds

$$Q_n(\alpha, \beta) := P_n(t, s).$$

They are of degree n, counting α to be of degree 1, and β of degree 2. Thus, we have justified the main result of the paper.

Theorem 5.1. *The Wallis power function has the asymptotic expansion of the following form:*

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \sim x + \sum_{n=0}^{\infty} Q_{n+1}(\alpha,\beta)x^{-n}, \tag{5.1}$$

where it holds $Q_0(\alpha,\beta) = 1$ *and*

$$Q_n(\alpha,\beta) = \frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\nabla_k(\alpha,\beta)Q_{n-k}(\alpha,\beta), \quad n \ge 1. \tag{5.2}$$

Using this formula we can get the first few polynomials (Q_n):

$$\begin{aligned}
Q_1 &= \alpha\\
Q_2 &= \frac{1}{6}\beta\\
Q_3 &= -\frac{1}{6}\alpha\beta\\
Q_4 &= \frac{1}{6}\alpha^2\beta - \frac{1}{60}\beta - \frac{13}{360}\beta^2\\
Q_5 &= -\frac{1}{6}\alpha^3\beta + \frac{1}{20}\alpha\beta + \frac{13}{120}\alpha\beta^2\\
Q_6 &= \frac{1}{6}\alpha^4\beta - \frac{1}{10}\alpha^2\beta - \frac{13}{60}\alpha^2\beta^2 + \frac{1}{126}\beta + \frac{53}{2520}\beta^2 + \frac{737}{45\,360}\beta^3.
\end{aligned} \tag{5.3}$$

This values suggest that the following theorems are valid.

Theorem 5.2. *Polynomial Q_n is divisible by β for every $n \ge 2$.*

Proof. It holds $Q_2 = \frac{\beta}{6}$. The theorem can be proved by induction.

Let us suppose that $Q_2, \dots, Q_{n-1}$ are divisible by β. According to formula (5.2), it is sufficient to show that

$$\alpha\nabla_{n-1} - \nabla_n$$

is divisible by β. This expression is equal to

$$\frac{t+s-1}{2}\cdot\frac{B_n(t)-B_n(s)}{n(t-s)} - \frac{B_{n+1}(t)-B_{n+1}(s)}{(n+1)(t-s)}.$$

From the property of the Bernoulli polynomials

$$B_n(t+1) - B_n(t) = nt^{n-1},$$

it follows that this expression vanishes if $t = s+1$ or $s = t+1$ and the theorem is proved. □

Theorem 5.3. *The sequence (Q_n) satisfies the following relation*

$$\frac{\partial Q_n}{\partial\alpha} = -(n-2)Q_{n-1} \tag{5.4}$$

for all $n \ge 2$.

Proof. Let us differentiate the main identity

$$F(x,\alpha,\beta) = \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{\frac{1}{t-s}} \sim x + \sum_{k=0}^{\infty} Q_{k+1}(\alpha,\beta)\frac{1}{x^k}$$

with respect to variable α. We get

$$\frac{\partial F}{\partial\alpha} = F\cdot\left[\log W\cdot\frac{\partial}{\partial\alpha}\left(\frac{1}{t-s}\right) + \frac{1}{t-s}\cdot\frac{1}{W}\cdot\frac{\partial W}{\partial\alpha}\right].$$

Since $\frac{\partial t}{\partial\alpha} = \frac{\partial s}{\partial\alpha} = 1$, we have

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{t-s}\right) = 0$$

and

$$\frac{\partial F}{\partial \alpha} = F \cdot \frac{\psi(x+t) - \psi(x+s)}{t-s}.$$

Both functions of the right-hand side have asymptotic expansion as $x \to \infty$. Therefore asymptotic expansion for $\partial F/\partial\alpha$ exists and can be obtained by termwise differentiation:

$$F \cdot \frac{\psi(x+t) - \psi(x+s)}{t-s} \sim \sum_{k=0}^{\infty} \frac{\partial}{\partial \alpha} Q_{k+1}(\alpha, \beta) \frac{1}{x^k}.$$

From relation (2.7)

$$1 + \sum_{k=1}^{\infty} Q_{k+1}(\alpha, \beta) \frac{-k}{x^{k+1}} \sim \sum_{k=0}^{\infty} \frac{\partial}{\partial \alpha} Q_{k+1}(\alpha, \beta) \frac{1}{x^k}$$

wherefrom it follows (5.4). □

Let

$$A_n(\beta) := Q_n(0, \beta). \tag{5.5}$$

It is convenient to calculate sequence (Q_n) by the following formula

$$Q_n(\alpha, \beta) = -(n-2) \int_0^{\alpha} Q_{n-1}(\alpha, \beta) \mathrm{d}\alpha + A_n(\beta). \tag{5.6}$$

It holds $Q_0 = 1$, $Q_1 = \alpha$, hence $A_0 = 1$, $A_1 = 0$. From formula (5.2) and Theorem 4.4 we obtain the following algorithm.

Theorem 5.4. *The sequence (A_n) satisfies relation*

$$\begin{aligned} A_{2n+1}(\beta) &= 0, \\ A_{2n}(\beta) &= -\frac{1}{2n} \sum_{k=1}^{n} \frac{1}{2k+1} G_k(\beta) A_{2n-2k}(\beta). \end{aligned} \tag{5.7}$$

Proof. We need to prove the first statement. For odd n, indices k and $n-k$ have different parity. Using relations (4.9) and (5.2) we conclude that all summands in (5.7) vanish. The rest follows from Theorem 4.4. □

Here are the first few polynomials (A_n):

$$\begin{aligned} A_0 &= 1 \\ A_2 &= \frac{1}{6}\beta \\ A_4 &= -\frac{1}{60}\beta - \frac{13}{360}\beta^2 \\ A_6 &= \frac{1}{126}\beta + \frac{53}{2520}\beta^2 + \frac{737}{45\,360}\beta^3 \\ A_8 &= -\frac{1}{120}\beta - \frac{3559}{151\,200}\beta^2 - \frac{3509}{151\,200}\beta^3 - \frac{50\,801}{5\,443\,200}\beta^4. \end{aligned}$$

6. Asymptotic expansion of the Wallis quotient

The quotient of gamma functions $\Gamma(x+t)/\Gamma(x+s)$ has been studied a long time ago; see e.g. [14]. There are two approaches. The first one leads to the expansion through variable x; the known formula (1.5) is proved in [14]. This leads to the approximation of the type:

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim \sum_{n=0}^{\infty} C_n(t, s) x^{t-s-n},$$

where the leading coefficients are

$$\begin{aligned} C_0 &= 1, \\ C_1 &= \frac{1}{2}(t-s)(t+s-1), \\ C_2 &= \frac{1}{12}\binom{t-s}{2}[3(t+s-1)^2 - t + s - 1]. \end{aligned} \tag{6.1}$$

The general formula of this type includes the generalized Bernoulli polynomials. Using same methods as in the previous results, another expansion for the Wallis quotient can be derived, but without generalized Bernoulli polynomials.

Theorem 6.1. *It holds*

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim \sum_{n=0}^{\infty} C_n(t,s) x^{t-s-n}, \tag{6.2}$$

where polynomials $C_n(t,s)$ are defined by

$$\begin{aligned} C_0(t,s) &= 1, \\ C_n(t,s) &= \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} \frac{B_{k+1}(t) - B_{k+1}(s)}{k+1} C_{n-k}(t,s) \end{aligned} \tag{6.3}$$

and $B_k(t)$ stands for the Bernoulli polynomials.

Proof. Since it holds

$$\frac{\partial}{\partial x} W(x,t,s) = W(x,t,s)[\psi(x+t) - \psi(x+s)]$$

and the functions on the right-hand side have asymptotic expansion as $x \to \infty$, (6.2) can be differentiated by terms. Therefore,

$$\begin{aligned} W(x,t,s)[\psi(x+t) - \psi(x+s)] &\sim x^{t-s} \sum_{n=1}^{\infty} (-n) C_n(t,s) x^{-n-1} + (t-s) x^{t-s-1} \sum_{n=0}^{\infty} C_n(t,s) x^{-n} \\ &\sim x^{t-s} \sum_{n=1}^{\infty} (-n) C_n(t,s) x^{-n-1} + (t-s) \frac{1}{x} \cdot W(x,t,s). \end{aligned}$$

Hence

$$W(x,t,s) \left[\psi(x+t) - \psi(x+s) - (t-s)\frac{1}{x} \right] \sim x^{t-s} \sum_{n=1}^{\infty} (-n) C_n(t,s) x^{-n-1}$$

wherefrom it follows, applying (1.4)

$$\left(\sum_{n=0}^{\infty} C_n x^{-n} \right) \left(\sum_{k=1}^{\infty} (-1)^k \frac{B_{k+1}(t) - B_{k+1}(s)}{k+1} x^{-k-1} \right) \sim - \sum_{n=1}^{\infty} n C_n x^{-n-1}.$$

After rearrangement of the terms, we have

$$-nC_n = \sum_{k=1}^{n} (-1)^k \frac{B_{k+1}(t) - B_{k+1}(s)}{k+1} C_{n-k}$$

and the proof is complete. □

The second approach to the asymptotic expansion of the Wallis quotient is through shifted variable

$$w = x + t - \rho,$$

where $2\rho = t - s + 1$, see [9,15, p. 34]

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim w^{t-s} \sum_{n=0}^{\infty} \frac{B_{2n}^{(2\rho)}(\rho)(t-s)_{2n}}{(2n)!} w^{-2n}. \tag{6.4}$$

It is obvious that this expansion is more naturally connected with the Wallis ratio. But, the coefficients $B_{2k}^{(2\rho)}(\rho)$ are somewhat strange here. Namely, $a \mapsto B_{2n}^{(a)}(x)$ and $x \mapsto B_{2n}^{(a)}(x)$ are polynomials of order $2n$, and $\rho \mapsto B_{2n}^{(2\rho)}(\rho)$ is a polynomial of order n. This leads to the conclusion that there exists a more natural expansion of the type (1.5). We shall show that this expansion contains only the "usual" Bernoulli polynomials.

Theorem 6.2. *The Wallis quotient has expansion of the type*

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim \sum_{n=0}^{\infty} S_{2n}(t-s,\beta) w^{t-s-2n}, \tag{6.5}$$

where $w = x + \alpha$. The coefficients S_{2n} satisfy

$$S_0 = 1,$$
$$S_{2n} = \frac{1}{2n} \sum_{k=1}^{n} \frac{1}{2k+1} (t-s) G_k(\beta) S_{2n-2k}, \tag{6.6}$$

and they depend on $t - s$ and β.

Proof. We shall use (6.2) and (6.3). First, note that it holds

$$w = x + t - \rho = x + \frac{t+s-1}{2} = x + \alpha$$

and

$$x + t = x + \alpha + \beta_1, \qquad x + s = x + \alpha + \beta_2.$$

Therefore,

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} = \frac{\Gamma(w+\beta_1)}{\Gamma(w+\beta_2)} \sim \sum_{n=0}^{\infty} C_n(\beta_1, \beta_2) w^{t-s-n},$$

where

$$C_n(\beta_1, \beta_2) = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} C_{n-k}(\beta_1, \beta_2) \frac{B_{k+1}(\beta_1) - B_{k+1}(\beta_2)}{k+1}.$$

Since it holds $\beta_1 + \beta_2 = 1$, properties of the Bernoulli polynomials can be used.

$$B_{2n}(\beta_1) = B_{2n}(\beta_2),$$
$$B_{2n+1}(\beta_1) = -B_{2n+1}(\beta_2).$$

It is easy to see that odd members of the sequence (S_n) vanish and for even members, we can use Theorem 3.2. It follows that coefficients depend only on $t - s$ and β and (6.6) holds true. □

7. Asymptotic expansions through shifted variable

We shall consider now the asymptotic expansion of the Wallis power function of the second kind. It will be shown that this approach is the most natural and coefficients in this expansion will depend only on variable β.

Theorem 7.1. *The Wallis power function has the expansion of the type*

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} \sim \sum_{n=0}^{\infty} A_{2n}(\beta) w^{-2n+1} \tag{7.1}$$

where $w = x + \alpha$ and $A_{2n}(\beta)$ are defined by (5.7).

Proof. The Wallis power function can be written in a way

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} = \left[\frac{\Gamma(w+\beta_1)}{\Gamma(w+\beta_2)} \right]^{1/(\beta_1-\beta_2)}.$$

One can apply asymptotic expansion to the function on the right. But, for the corresponding variables α', β' in the expansion (5.1) it holds

$$\alpha' = \frac{\beta_1 + \beta_2 - 1}{2} = 0,$$
$$\beta' = \frac{1}{4}[1 - (\beta_1 - \beta_2)^2] = \frac{1}{4}[1 - (t-s)^2] = \beta.$$

Therefore, coefficients $Q_n(\alpha', \beta')$ in this expansion are equal to $Q_n(0, \beta) = A_n(\beta)$. □

The theorem just proved enables us to derive closed form of the polynomials Q_n.

Theorem 7.2. *The Wallis power function has the asymptotic expansion*

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \sim x + \sum_{n=0}^{\infty} Q_{n+1}(\alpha,\beta)x^{-n}, \tag{7.2}$$

where

$$Q_n(\alpha,\beta) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{1-2k}{n-2k} A_{2k}(\beta)\alpha^{n-2k} \tag{7.3}$$

and $A_{2n}(\beta)$ are defined by (5.7).

Proof. Let us write a sequence of asymptotic expansions

$$\begin{aligned}
F(x,t,s) &\sim w + \sum_{n=1}^{\infty} A_{2n}(\beta) w^{-2n+1} \\
&\sim w + \sum_{n=1}^{\infty} A_{2n}(\beta)\frac{1}{x^{2n-1}}\left(1+\frac{\alpha}{x}\right)^{-2n+1} \\
&\sim w + \sum_{n=1}^{\infty} A_{2n}(\beta)\frac{1}{x^{2n-1}}\left[\sum_{k=0}^{\infty}\binom{-2n+1}{k}\frac{\alpha^k}{x^k}\right] \\
&\sim w + \sum_{n=2}^{\infty}\left[\sum_{k=1}^{\lfloor n/2\rfloor}\binom{-2k+1}{n-2k}A_{2k}(\beta)\alpha^{n-2k}\right]\frac{1}{x^{n-1}}.
\end{aligned}$$

On the other hand, it holds

$$F(x,t,s) \sim x + \alpha + \sum_{n=2}^{\infty} Q_n(\alpha,\beta)x^{-n+1}$$

and the theorem is proved. □

8. Some applications

In this section, we shall apply our theorems to improve some known results. We shall restrict ourself only to the direct consequences of derived formulas. More detailed applications are postponed to another papers; see [16,17].

Using (2.9) and calculating $Q_k(\alpha,\beta)$ as described in Section 5, we get the following result which improves formula (1.9).

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{\frac{1}{t-s}} \approx x + \sum_{n=0}^{5} Q_n(\alpha,\beta)x^{-n}. \tag{8.1}$$

For $t = 1, s = \frac{1}{2}$, we have $\alpha = \frac{1}{4}, \beta = \frac{3}{16}$ and an improvement of the formula (1.8) from the recent paper [5] can be easily derived. For simplicity, we shall only give one more term in the expansion. Of course, using general formulas from the previous sections one can derive as many coefficients as necessary.

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt{n+\frac{1}{4}+\frac{1}{32n}-\frac{1}{128n^2}-\frac{5}{2048n^3}+\frac{23}{8192n^4}+\frac{53}{65\,536n^5}}. \tag{8.2}$$

If we choose $t = \frac{3}{4}, s = \frac{1}{4}$, the following result is obtained:

$$\frac{\Gamma\left(n+\frac{3}{4}\right)}{\Gamma\left(n+\frac{1}{4}\right)} \approx \sqrt{n+\frac{1}{32n}-\frac{9}{2048n^3}+\frac{153}{65\,536n^5}-\frac{21\,429}{8\,388\,608n^7}}. \tag{8.3}$$

The previous result can be used with shifted variable $x = n + \frac{1}{4}$ as in Theorem 7.1 to get better result for a "n and a quarter" formula (see [10]):

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \approx \sqrt{\left(n+\frac{1}{4}\right)+\frac{1}{32\left(n+\frac{1}{4}\right)}-\frac{9}{2048\left(n+\frac{1}{4}\right)^3}+\frac{153}{65\,536\left(n+\frac{1}{4}\right)^5}}. \tag{8.4}$$

Leading binomial coefficient $\binom{2n}{n}$ is connected with the Wallis quotient through the duplication formula [12, p. 256]:

$$\binom{2n}{n} = \frac{\Gamma(2n+1)}{\Gamma(n+1)^2} = \frac{2}{n} \cdot \frac{2^{2n-1}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)}.$$

The expansion for the last fraction can be obtained from (2.1) taking $t = \frac{1}{2}$, $s = 0$. But in this case, we have $\alpha = -\frac{1}{4}$, $\beta = \frac{3}{16}$ and the required expansion can be written immediately using (8.2). Note that, because of (7.3), sign should be changed to members with odd index:

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)} \approx \sqrt{n - \frac{1}{4} + \frac{1}{32n} + \frac{1}{128n^2} - \frac{5}{2048n^3} - \frac{23}{8192n^4} + \frac{53}{65\,536n^5}}. \tag{8.5}$$

Therefore,

$$\binom{2n}{n} \approx \frac{2^{2n}}{\sqrt{n\pi}} \sqrt{1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{1}{128n^3} - \frac{5}{2048n^4} - \frac{23}{8192n^5} + \frac{53}{65\,536n^6}}. \tag{8.6}$$

The well-known asymptotic expansion for the central binomial coefficient has similar form, see e.g. [9, p. 35],

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{n\pi}} \left[1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} - \frac{21}{32\,768n^4} + \cdots\right]. \tag{8.7}$$

This expansion will be investigated in the forthcoming paper; see [16].

More useful quotients of the Wallis type can be easily derived from the general formula. For example, the following ones are connected with central polynomial coefficients:

$$\frac{\Gamma\left(n+\frac{1}{3}\right)}{\Gamma(n)} \sim \left[n - \frac{1}{3} + \frac{1}{27n} + \frac{1}{81n^2} - \frac{1}{729n^3} - \frac{1}{243n^4} - \frac{13}{59\,049n^5} + \cdots\right]^{1/3}, \tag{8.8}$$

$$\frac{\Gamma(n+\frac{1}{r})}{\Gamma(n-\frac{1}{r})} \sim \left[n - \frac{1}{2} + \frac{r^2-4}{24r^2n} + \frac{r^2-4}{48r^2n^2} + \frac{23r^4-40r^2-208}{5760r^4n^3} + \cdots\right]^{2/r}. \tag{8.9}$$

Some other formulas concerning various approximation of binomial coefficients can be derived using expansions given in the present paper. See [16] for a detailed explanation, where also the asymptotic expansion for the truncation of duplication formula for gamma function is discussed.

Application of our method also leads to the new asymptotic expansions of the gamma function which improve many known approximation formulas for the factorial function; see [17].

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